Solution: a. The exponential growth function is $y = f(t) = ab^t$, where $a = 2000$ because the initial population is 2000 squirrels

The annual growth rate is 3% per year, stated in the problem. We will express this in decimal form as $r = 0.03$

Then $b = 1+r = 1+0.03 = 1.03$

Answer: The exponential growth function is $y = f(t) = 2000(1.03^t)$

b. After 5 years, the squirrel population is $y = f(5) = 2000(1.03^5) \approx 2319$ squirrels

After 10 years, the squirrel population is $y = f(10) = 2000(1.03^{10}) \approx 2688$ squirrels

◆***Example 3*** A large lake has a population of 1000 frogs. Unfortunately the frog population is decreasing at the rate of 5% per year.
Let t = number of years and $y = g(t)$ = the number of frogs in the lake at time t.
a. Find the exponential decay function that models the population of frogs.
b. Calculate the size of the frog population after 10 years.

Solution: a. The exponential decay function is $y = g(t) = ab^t$, where $a = 1000$ because the initial population is 1000 frogs

The annual decay rate is 5% per year, stated in the problem. The words decrease and decay indicated that r is negative. We express this as $r = -0.05$ in decimal form.

Then, $b = 1+ r = 1+ (-0.05) = 0.95$

Answer: The exponential decay function is: $y = g(t) = 1000(0.95^t)$

b. After 10 years, the frog population is $y = g(10) = 1000(0.95^{10}) \approx 599$ frogs

◆***Example 4*** A population of bacteria is given by the function $y = f(t) = 100(2^t)$, where t is time measured in hours and y is the number of bacteria in the population.
a. What is the initial population?
b. What happens to the population in the first hour?
c. How long does it take for the population to reach 800 bacteria?

Solution: a. The initial population is 100 bacteria. We know this because a = 100 and because at time t = 0, then $f(0) = 100(2^0) = 100(1)=100$

b. At the end of 1 hour, the population is $y = f(1) = 100(2^1) = 100(2)=200$ bacteria. The population has doubled during the first hour.

c. We need to find the time t at which f(t) = 800. Substitute 800 as the value of y:

$$y = f(t) =100(2^t)$$

$$800 =100(2^t)$$

Divide both sides by 100 to isolate the exponential expression on the one side

$$8 = 1(2^t)$$

$8 = 2^3$, so it takes t = 3 hours for the population to reach 800 bacteria.

EXPRESSING EXPONENTIAL FUNCTIONS IN THE FORMS $y = ab^t$ *and* $y = ae^{kt}$

Now that we've developed our equation solving skills, we revisit the question of expressing exponential functions equivalently in the forms $y = ab^t$ and $y = ae^{kt}$

We've already determined that if given the form $y = ae^{kt}$, it is straightforward to find b.

◆***Example 7*** For the following examples, assume t is measured in years.

a. Express $y = 3500\, e^{0.25t}$ in form $y = ab^t$ and find the annual percentage growth rate.

b. Express $y = 28000\, e^{-0.32t}$ in form $y = ab^t$ and find the annual percentage decay rate.

Solution: a. Express $y = 3500\, e^{0.25t}$ in the form $y = ab^t$

$y = ae^{kt} = ab^t$

$a(e^k)^t = ab^t$

Thus $e^k = b$

In this example $b = e^{0.25} \approx 1.284$

We rewrite the growth function as $y = 3500(1.284^t)$

To find r, recall that $b = 1+r$

$1.284 = 1+r$

$0.284 = r$

The continuous growth rate is k = 0.25 and the annual percentage growth rate is 28.4% per year.

b. Express $y = 28000\, e^{-0.32t}$ in the form $y = ab^t$

$y = ae^{kt} = ab^t$

$a(e^k)^t = ab^t$

Thus $e^k = b$

In this example $b = e^{-0.32} \approx 0.7261$

We rewrite the growth function as $y = 28000(0.7261^t)$

To find r, recall that $b = 1+r$

$0.7261 = 1+r$

$0.2739 = r$

The continuous decay rate is k = –0.32 and the annual percentage decay rate is 27.39% per year.

In the sentence, we omit the negative sign when stating the annual percentage decay rate because we have used the word "decay" to indicate that r is negative.

SECTION 5.3 PROBLEM SET: LOGARITHMS AND LOGARITHMIC FUNCTIONS

Rewrite each of these exponential expressions in logarithmic form:

1) $3^4 = 81$	2) $10^5 = 100{,}000$
3) $5^{-2} = 0.04$	4) $4^{-1} = 0.25$
5) $16^{1/4} = 2$	6) $9^{1/2} = 3$

Rewrite each of these logarithmic expressions in exponential form:

7) $\log_5 625 = 4$	8) $\log_2 (1/32) = -5$
9) $\log_{11} 1331 = 3$	10) $\log_{10} 0.0001 = -4$
11) $\log_{64} 4 = 1/3$	12) $\ln\sqrt{e} = \frac{1}{2}$

If the expression is in exponential form, rewrite it in logarithmic form.
If the expression is in logarithmic form, rewrite it in exponential form.

13) $5^x = 15625$	14) $x = 9^3$
15) $\log_5 125 = x$	16) $\log_3 x = 5$
17) $\log_{10} y = 4$	18) $e^x = 10$
19) $\ln x = -1$	20) $e^5 = y$

6.4 Present Value of an Annuity and Installment Payment

In this section, you will learn to:

1. *Find the present value of an annuity.*
2. *Find the amount of installment payment on a loan.*

PRESENT VALUE OF AN ANNUITY

In Section 6.2, we learned to find the future value of a lump sum, and in Section 6.3, we learned to find the future value of an annuity. With these two concepts in hand, we will now learn to amortize a loan, and to find the present value of an annuity.

The **present value** of an annuity is the amount of money we would need now in order to be able to make the payments in the annuity in the future. In other word, the present value is the value now of a future stream of payments.

We start by breaking this down step by step to understand the concept of the present value of an annuity. After that, the examples provide a more efficient way to do the calculations by working with concepts and calculations we have already explored in Sections 6.2 and 6.3.

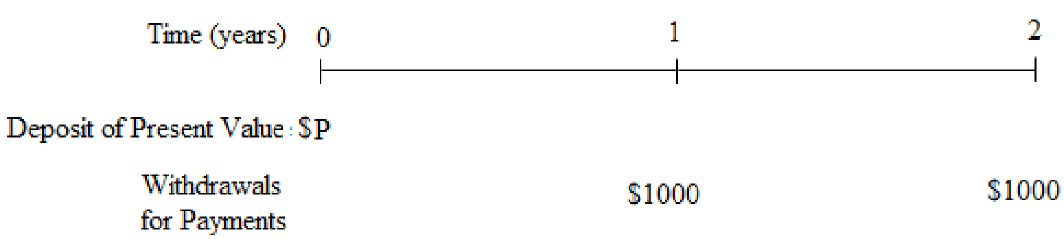
Suppose Carlos owns a small business and employs an assistant manager to help him run the business. Assume it is January 1 now. Carlos plans to pay his assistant manager a $1000 bonus at the end of this year and another $1000 bonus at the end of the following year. Carlos' business had good profits this year so he wants to put the money for his assistant's future bonuses into a savings account now. The money he puts in now will earn interest at the rate of 4% per year compounded annually while in the savings account.

How much money should Carlos put into the savings account now so that he will be able to withdraw $1000 one year from now and another $1000 two years from now?

At first, this sounds like a sinking fund. But it is different. In a sinking fund, we put money into the fund with periodic payments to save to accumulate to a specified lump sum that is the future value at the end of a specified time period.

In this case we want to put a lump sum into the savings account now, so that lump sum is our principal, P. Then we want to withdraw that amount as a series of period payments; in this case the withdrawals are an annuity with $1000 payments at the end of each of two years.

We need to determine the amount we need in the account now, the present value, to be able to make withdraw the periodic payments later.



We use the compound interest formula from Section 6.2 with r = 0.04 and n = 1 for annual compounding to determine the present value of each payment of $1000.

Consider the first payment of $1000 at the end of year 1. Let P_1 be its present value

$\$1000 = P_1(1.04)^1$ so P_1=$961.54

Now consider the second payment of $1000 at the end of year 2. Let P_2 is its present value

$\$1000 = P_2(1.04)^2$ so P_2=$924.56

To make the $1000 payments at the specified times in the future, the amount that Carlos needs to deposit now is the present value $P = P_1 + P_2 = \$961.54 + \$924.56 = \$1886.10$

The calculation above was useful to illustrate the meaning of the present value of an annuity. But it is not an efficient way to calculate the present value. If we were to have a large number of annuity payments, the step by step calculation would be long and tedious.

Example 1 investigates and develops an efficient way to calculate the present value of an annuity, by relating the future (accumulated) value of an annuity and its present value.

◆***Example 1*** Suppose you have won a lottery that pays $1,000 per month for the next 20 years. But, you prefer to have the entire amount now. If the interest rate is 8%, how much will you accept?

Solution: This classic present value problem needs our complete attention because the rationalization we use to solve this problem will be used again in the problems to follow.

Consider, for argument purposes, that two people Mr. Cash, and Mr. Credit have won the same lottery of $1,000 per month for the next 20 years. Mr. Credit is happy with his $1,000 monthly payment, but Mr. Cash wants to have the entire amount now.

Our job is to determine how much Mr. Cash should get. We reason as follows:

If Mr. Cash accepts P dollars, then the P dollars deposited at 8% for 20 years should yield the same amount as the $1,000 monthly payments for 20 years.
In other words, we are comparing the future values for both Mr. Cash and Mr. Credit, and we would like the future values to equal.

Since Mr. Cash is receiving a lump sum of x dollars, its future value is given by the lump sum formula we studied in Section 6.2, and it is

$$A = P(1 + .08/12)^{240}$$

Since Mr. Credit is receiving a sequence of payments, or an annuity, of $1,000 per month, its future value is given by the annuity formula we learned in Section 6.3. This value is

$$A = \frac{\$1000\,[(1 + .08/12)^{240} - 1]}{.08/12}$$

The only way Mr. Cash will agree to the amount he receives is if these two future values are equal. So we set them equal and solve for the unknown.

Finally, we note that many finite mathematics and finance books develop the formula for the present value of an annuity differently.

Instead of using the formula : $P(1 + r/n)^{nt} = \dfrac{m[(1 + r/n)^{nt} - 1]}{r/n}$ (Formula 6.4.1)

and solving for the present value P after substituting the numerical values for the other items in the formula, many textbooks first solve the formula for P in order to develop a new formula for the present value. Then the numerical information can be substituted into the present value formula and evaluated, without needing to solve algebraically for P.

Alternate Method to find Present Value of an Annuity

Starting with formula 6.4.1: $P(1 + r/n)^{nt} = \dfrac{m[(1 + r/n)^{nt} - 1]}{r/n}$

Divide both sides by $(1+r/n)^{nt}$ to isolate P, and simplify

$$P = \frac{m[(1+r/n)^{nt} - 1]}{r/n} \cdot \frac{1}{(1+r/n)^{nt}}$$

$$P = \frac{m[1-(1+r/n)^{-nt}]}{r/n}$$ (Formula 6.4.2)

The authors of this book believe that it is easier to use formula 6.4.1 at the top of this page and solve for P or m as needed. In this approach there are fewer formulas to understand, and many students find it easier to learn. In the problems the rest of this chapter, when a problem requires the calculation of the present value of an annuity, formula 6.4.1 will be used.

However, some people prefer formula 6.4.2, and it is mathematically correct to use that method. Note that if you choose to use formula 6.4.2, you need to be careful with the negative exponents in the formula. And if you needed to find the periodic payment, you would still need to do the algebra to solve for the value of m.

It would be a good idea to check with your instructor to see if he or she has a preference. In fact, you can usually tell your instructor's preference by noting how he or she explains and demonstrates these types of problems in class.

Tree diagrams help us visualize the different possibilities, but they are not practical when the possibilities are numerous. Besides, we are mostly interested in finding the number of elements in the set and not the actual list of all possibilities; once the problem is envisioned, we can solve it without a tree diagram. The two examples we just solved may have given us a clue to do just that.

Let us now try to solve Example 2 without a tree diagram. The problem involves three steps: choosing a blouse, choosing a skirt, and choosing a pair of pumps. The number of ways of choosing each are listed below. By multiplying these three numbers we get 12, which is what we got when we did the problem using a tree diagram.

The number of ways of choosing a blouse	The number of ways of choosing a skirt	The number of ways of choosing pumps
2	3	2

The procedure we just employed is called the multiplication axiom.

THE MULTIPLICATION AXIOM

If a task can be done in m ways, and a second task can be done in n ways, then the operation involving the first task followed by the second can be performed in $m \cdot n$ ways.

The general multiplication axiom is not limited to just two tasks and can be used for any number of tasks.

◆***Example 3*** A truck license plate consists of a letter followed by four digits. How many such license plates are possible?

Solution: Since there are 26 letters and 10 digits, we have the following choices for each.

Letter	Digit	Digit	Digit	Digit
26	10	10	10	10

Therefore, the number of possible license plates is $26 \cdot 10 \cdot 10 \cdot 10 \cdot 10 = 260{,}000$.

◆***Example 4*** In how many different ways can a 3-question true-false test be answered?

Solution: Since there are two choices for each question, we have

Question 1	Question 2	Question 3
2	2	2

Applying the multiplication axiom, we get $2 \cdot 2 \cdot 2 = 8$ different ways.

We list all eight possibilities: TTT, TTF, TFT, TFF, FTT, FTF, FFT, FFF

The reader should note that the first letter in each possibility is the answer corresponding to the first question, the second letter corresponds to the answer to the second question, and so on. For example, TFF, says that the answer to the first question is given as true, and the answers to the second and third questions false.

◆***Example 9*** A jar contains three marbles numbered 1, 2, and 3. If two marbles are drawn **with replacement**, what is the probability that the sum of the numbers is 5?

Note: The two marbles in this example are drawn consecutively **with replacement**. That means that after a marble is drawn it IS replaced in the jar, and therefore is available to select again on the second draw.

Solution: When two marbles are drawn with replacement, the sample space consists of the following nine possibilities.

S = {(1,1), (1, 2), (1, 3), (2, 1), (2,2), (2, 3), (3, 1), (3, 2), (3,3)}

Note that (1,1), (2,2) and (3,3) are listed in the sample space. These outcomes are possible when drawing with replacement, because once the first marble is drawn and replaced, that marble is not available in the jar to be drawn again.

Let the event E represent that the sum of the numbers is four. Then

E = {(2, 3), (3, 2) }

Therefore, the probability of F is P(E) = 2/9

Note that in Example 9 when we selected marbles with replacement, the probability has changed from Example 7 where we selected marbles without replacement.

◆***Example 10*** A jar contains three marbles numbered 1, 2, and 3. If two marbles are drawn **with replacement**, what is the probability that the sum of the numbers is *at least* 4?

Solution: The sample space when drawing with replacement consists of the following nine possibilities.

S = {(1,1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3,3)}

Let the event F represent that the sum of the numbers is at least four. Then

F = {(1, 3), (3, 1), (2, 3), (3, 2), (2,2), (3,3)}

Therefore, the probability of F is

P(F) = 6/9 or 2/3.

Note that in Example 10 when we selected marbles with replacement, the probability is the same as in Example 8 where we selected marbles without replacement.

Thus sampling with or without replacement MAY change the probabilities, but may not, depending on the situation in the particular problem under consideration. We'll re-examine the concepts of sampling with and without replacement in Section 8.3.

◆***Example 11*** One 6 sided die is rolled once. Find the probability that the result is greater than 4.

Solution: The sample space consists of the following six possibilities in set S: S={1,2,3,4,5,6}

Let E be the event that the number rolled is greater than four: E={5,6}

Therefore, the probability of E is: P(E) = 2/6 or 1/3.

SECTION 8.1 PROBLEM SET: SAMPLE SPACES AND PROBABILITY

For problems 23 – 27: Two dice are rolled. Find the following probabilities.

23) P(the sum of the dice is 5)	24) P(the sum of the dice is 8)
25) P(the sum is 3 or 6)	26) P(the sum is more than 10)
27) P(the result is a double) (Hint: a double means that both dice show the same value)	

For problems 28-31: A jar contains four marbles numbered 1, 2, 3, and 4. Two marbles are drawn randomly WITHOUT REPLACEMENT. That means that after a marble is drawn it is NOT replaced in the jar before the second marble is selected. Find the following probabilities.

28) P(the sum of the numbers is 5)	29) P(the sum of the numbers is odd)
30) P(the sum of the numbers is 9)	31) P(one of the numbers is 3)

For problems 32-33: A jar contains four marbles numbered 1, 2, 3, and 4. Two marbles are drawn randomly WITH REPLACEMENT. That means that after a marble is drawn it is replaced in the jar before the second marble is selected. Find the following probabilities.

32) P(the sum of the numbers is 5)	33) P(the sum of the numbers is 2)

◆***Example 3*** A family has three children. Determine whether the following pair of events are mutually exclusive.

M = {The family has at least one boy}   N = {The family has all girls}

Solution: Although the answer may be clear, we list both the sets.

M = {BBB, BBG, BGB, BGG, GBB, GBG, GGB} and N = {GGG}

Clearly, $M \cap N = \varnothing$

Therefore, events M and N are mutually exclusive.

We will now consider problems that involve the union of two events.

Given two events, E, F, then finding the probability of $E \cup F$, is the same as finding the probability that E will happen, or F will happen, or both will happen.

◆***Example 4*** If a die is rolled, what is the probability of obtaining an even number or a number greater than four?

Solution: Let E be the event that the number shown on the die is an even number, and let F be the event that the number shown is greater than four.

The sample space S = {1, 2, 3, 4, 5, 6}. The event E = {2, 4, 6}, and event F = {5, 6}

We need to find $P(E \cup F)$.

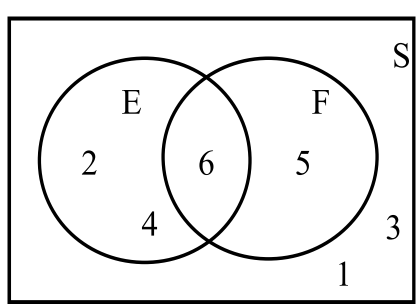
Since P(E) = 3/6, and P(F) = 2/6, a student may say $P(E \cup F)$ = 3/6 + 2/6. This will be incorrect because the element 6, which is in both E and F has been counted twice, once as an element of E and once as an element of F. In other words, the set $E \cup F$ has only four elements and not five: set $E \cup F$ = {2,4,5,6}

Therefore, $P(E \cup F)$ = 4/6 and not 5/6.

This can be illustrated by a Venn diagram. We'll use the Venn Diagram to re-examine Example 4 and derive a probability rule that we can use to calculate probabilities for unions of events.

The sample space S, the events E and F, and $E \cap F$ are listed below.

S = {1, 2, 3, 4, 5, 6}, E = {2, 4, 6}, F = {5, 6}, and $E \cap F$ = {6}.



The above figure shows S, E, F, and $E \cap F$.

In the second part of the above example, we were finding the probability of obtaining a king knowing that a face card had shown. This is an example of **conditional probability**. Whenever we are finding the probability of an event E under the condition that another event F has happened, we are finding conditional probability.

The symbol P(E | F) denotes the problem of finding the probability of E given that F has occurred. We read P(E | F) as "the probability of E, given F."

◆***Example 1*** A family has three children. Find the conditional probability of having two boys and a girl given that the first born is a boy.

Solution: Let event E be that the family has two boys and a girl, and F that the first born is a boy.

First, we the sample space for a family of three children as follows.

S = {BBB, BBG, BGB, BGG, GBB, GBG, GGB, GGG}

Since we know the first born is a boy, our possibilities narrow down to four outcomes: BBB, BBG, BGB, and BGG.

Among the four, BBG and BGB represent two boys and a girl.

Therefore, P(E | F} = 2/4 or 1/2.

◆***Example 2*** One six sided die is rolled once.

a. Find the probability that the result is even.

b. Find the probability that the result is even given that the result is greater than three.

Solution: The sample space is S = {1,2,3,4,5,6}

Let event E be that the result is even and T be that the result is greater than 3.

a. P(E) = 3/6 because E = {2,4,6}

b. Because T = {4,5,6}, we know that 1, 2, 3 cannot occur; only outcomes 4, 5, 6 are possible. Therefore of the values in E, only 4, 6 are possible.

Therefore, P(E|T} = 2/3

◆***Example 3*** A fair coin is tossed twice.

a. Find the probability that the result is is two heads.

b. Find the probability that the result is two heads given that at least one head is obtained.

Solution: The sample space is S = {HH, HT, TH, TT}

Let event E be that the two heads are obtained and F be at least one head is obtained

a. P(E) = 1/4 because E = {HH} and the sample space S has 4 outcomes.

b. F = {HH, HT, TH}. Since at least one head was obtained, TT did not occur. We are interested in the probability event E={HH} out of the 3 outcomes in the reduced sample space F.

Therefore, P(E|F} = 1/3

Whenever the probability of an event E is not affected by the occurrence of another event F, and vice versa, we say that the two events E and F are **independent.** This leads to the following definition.

> Two Events E and F are **independent** if and only if at least one of the following two conditions is true.
>
> **1. $P(E \mid F) = P(E)$ or 2. $P(F \mid E) = P(F)$**
>
> If the events are not independent, then they are dependent.
>
> If one of these conditions is true, then both are true.

We can use the definition of independence to determine if two events are independent.

We can use that definition to develop another way to test whether two events are independent.

Recall the conditional probability formula:

$$P(E \mid F) = \frac{P(E \cap F)}{P(F)}$$

Multiplying both sides by P(F), we get

$$P(E \cap F) = P(E \mid F)\,P(F)$$

Now if the two events are independent, then by definition

$$P(E \mid F) = P(E)$$

Substituting, $P(E \cap F) = P(E)\,P(F)$

We state it formally as follows.

> **Test For Independence**
>
> Two events E and F are independent if and only if
>
> **$P(E \cap F) = P(E)\,P(F)$**

In the Examples 3 and 4, we'll examine how to check for independence using both methods:

- Examine the probability of intersection of events to check whether $P(E \cap F) = P(E)P(F)$
- Examine conditional probabilities to check whether $P(E|F)=P(E)$ or $P(F|E)=P(F)$

We need to use only **one** of these methods. Both methods, if used properly, will always give results that are consistent with each other.

Use the method that seems easier based on the information given in the problem.

◆***Example 3*** The table below shows the distribution of color-blind people by gender.

	Male(M)	Female(F)	Total
Color-Blind(C)	6	1	7
Not Color-Blind(N)	46	47	93
Total	52	48	100

where M represents male, F represents female, C represents color-blind, and N represents not color-blind. Are the events color-blind and male independent?

Solution 1: According to the test for independence, C and M are independent if and only if $P(C \cap M) = P(C)P(M)$.

From the table: $P(C) = 7/100$, $P(M) = 52/100$ and $P(C \cap M) = 6/100$

So $P(C)\,P(M) = (7/100)(52/100) = .0364$

which is **not** equal to $P(C \cap M) = 6/100 = .06$

Therefore, the two events are not independent. We may say they are dependent.

Solution 2: C and M are independent if and only if $P(C|M) = P(C)$.

From the total column $P(C) = 7/100 = 0.07$

From the male column $P(C|M) = 6/52 = 0.1154$

Therefore $P(C|M) \neq P(C)$, indicating that the two events are not independent.

◆***Example 4*** In a city with two airports, 100 flights were surveyed. 20 of those flights departed late.
45 flights in the survey departed from airport A; 9 of those flights departed late.
55 flights in the survey departed from airport B; 11 flights departed late.
Are the events "depart from airport A" and "departed late" independent?

Solution 1: Let A be the event that a flight departs from airport A, and L the event that a flight departs late. We have

$$P(A \cap L) = 9/100, \quad P(A) = 45/100 \text{ and } P(L) = 20/100$$

In order for two events to be independent, we must have $P(A \cap L) = P(A)\,P(L)$

Since $P(A \cap L) = 9/100 = 0.09$

and $P(A)\,P(L) = (45/100)(20/100) = 900/10000 = 0.09$

the two events "departing from airport A" and "departing late" are independent.

Solution 2: The definition of independent events states that two events are independent if $P(E|F)=P(E)$.

In this problem we are given that

$$P(L|A) = 9/45 = 0.2 \qquad \text{and} \qquad P(L) = 20/100 = 0.2$$

$P(L|A) = P(L)$, so events "departing from airport A" and "departing late" are independent.

◆***Example 7*** Given $P(B \mid A) = .4$. If A and B are independent, find P(B).

Solution: If A and B are independent, then by definition $P(B \mid A) = P(B)$

Therefore, $P(B) = .4$

◆***Example 8*** Given $P(A) = .7$, $P(B \mid A) = .5$. Find $P(A \cap B)$.

Solution 1: By definition $P(B \mid A) = \dfrac{P(A \cap B)}{P(A)}$

Substituting, we have

$$.5 = \frac{P(A \cap B)}{.7}$$

Therefore, $P(A \cap B) = .35$

Solution 2: Again, start with $P(B \mid A) = \dfrac{P(A \cap B)}{P(A)}$

Multiplying both sides by P(A) gives

$$P(A \cap B) = P(B \mid A)\, P(A) = (.5)(.7) = .35$$

Both solutions to Example 8 are actually the same, except that in Solution 2 we delayed substituting the values into the equation until after we solved the equation for $P(A \cap B)$. That gives the following result:

Multiplication Rule for events that are NOT independent

If events E and F are not independent

$\mathbf{P(E \cap F) = P(E \mid F)\, P(F)}$ **and** $\mathbf{P(E \cap F) = P(F \mid E)\, P(E)}$

◆***Example 9*** Given $P(A) = .5$, $P(A \cup B) = .7$, if A and B are independent, find P(B).

Solution: The addition rule states that

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Since A and B are independent, $P(A \cap B) = P(A)\, P(B)$

We substitute for $P(A \cap B)$ in the addition formula and get

$$P(A \cup B) = P(A) + P(B) - P(A)\, P(B)$$

By letting $P(B) = x$, and substituting values, we get

$$.7 = .5 + x - .5x$$

$$.7 = .5 + .5x$$

$$.2 = .5x$$

$$.4 = x$$

Therefore, $P(B) = .4$

SECTION 8.5 PROBLEM SET: INDEPENDENT EVENTS

17) $P(K|D) = 0.7$, $P(D) = 0.25$ and $P(K)=0.7$

a. Are events K and D independent?
Use probabilities to justify your conclusion.

b. Find $P(K \cap D)$

18) $P(R|S) = 0.4$, $P(S) = 0.2$ and $P(R)=0.3$

a. Are events R and S independent?
Use probabilities to justify your conclusion.

b. Find $P(R \cap S)$

19) At a college:
54% of students are female
25% of students are majoring in engineering.
15% of female students are majoring in engineering.

Event E = student is majoring in engineering
Event F = student is female

a. Are events E and F independent?
Use probabilities to justify your conclusion.

b. Find $P(E \cap F)$

20) At a college:
54% of all students are female
60% of all students receive financial aid.
60% of female students receive financial aid.

Event A = student receives financial aid
Event F = student is female

a. Are events A and F independent?
Use probabilities to justify your conclusion.

b. Find $P(A \cap F)$

9.2 Bayes' Formula

In this section you will learn to:

1. *find probabilities using Bayes' formula*
2. *use a probability tree to find and represent values needed when using Bayes' formula.*

In this section, we will develop and use Bayes' Formula to solve an important type of probability problem. Bayes' formula is a method of calculating the conditional probability P(F | E) from P(E | F). The ideas involved here are not new, and most of these problems can be solved using a tree diagram. However, Bayes' formula does provide us with a tool with which we can solve these problems without a tree diagram.

We begin with an example.

◆***Example 1*** Suppose you are given two jars. Jar I contains one black and 4 white marbles, and Jar II contains 4 black and 6 white marbles. If a jar is selected at random and a marble is chosen,

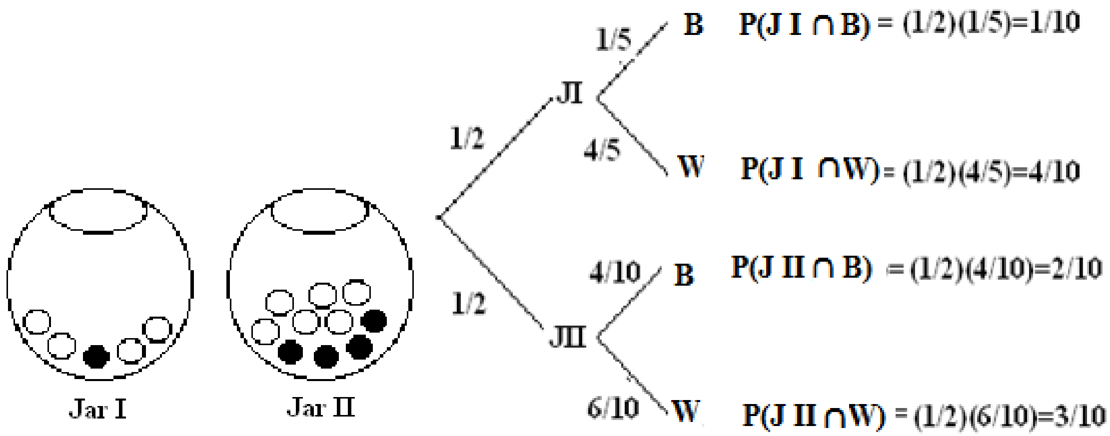
a. What is the probability that the marble chosen is a black marble?

b. If the chosen marble is black, what is the probability that it came from Jar I?

c. If the chosen marble is black, what is the probability that it came from Jar II?

Solution: Let J I be the event that Jar I is chosen, J II be the event that Jar II is chosen, B be the event that a black marble is chosen and W the event that a white marble is chosen.

We illustrate using a tree diagram.



a. The probability that a black marble is chosen is P(B) = 1/10 + 2/10 = 3/10.

b. To find P(J I | B), we use the definition of conditional probability, and we get

$$P(J\ I \mid B) = \frac{P(J\ I \cap B)}{P(B)} = \frac{1/10}{3/10} = \frac{1}{3}$$

c. Similarly, $P(J\ II \mid B) = \frac{P(J\ II \cap B)}{P(B)} = \frac{2/10}{3/10} = \frac{2}{3}$

The following is the generalization of Bayes' formula for n partitions.

Let S be a sample space that is divided into n partitions, $A_1, A_2, \ldots A_n$. If E is any event in S, then

$$\mathbf{P(A_i \mid E) = \frac{P(A_i)\, P(E \mid A_i)}{P(A_1)\, P(E \mid A_1) + P(A_2)\, P(E \mid A_2) + \cdots + P(A_n)\, P(E \mid A_n)}}$$

We begin with the following example.

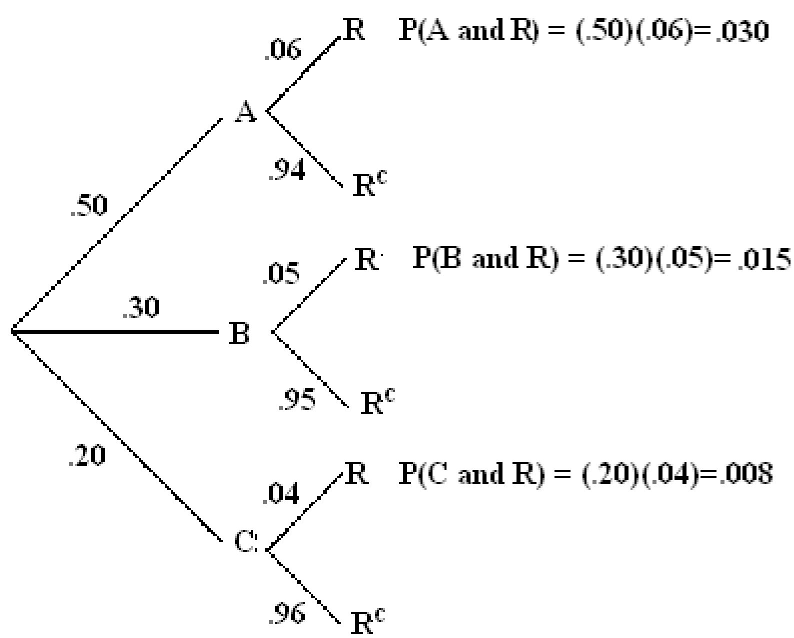
◆***Example 2*** A department store buys 50% of its appliances from Manufacturer A, 30% from Manufacturer B, and 20% from Manufacturer C. It is estimated that 6% of Manufacturer A's appliances, 5% of Manufacturer B's appliances, and 4% of Manufacturer C's appliances need repair before the warranty expires. An appliance is chosen at random. If the appliance chosen needed repair before the warranty expired, what is the probability that the appliance was manufactured by Manufacturer A? Manufacturer B? Manufacturer C?

Solution: Let A, B and C be the events that the appliance is manufactured by Manufacturer A, Manufacturer B, and Manufacturer C, respectively. Further, suppose that the event R denotes that the appliance needs repair before the warranty expires.

We need to find P(A | R), P(B | R) and P(C | R).

We will do this problem both by using a tree diagram and by using Bayes' formula.

We draw a tree diagram.



CHAPTER 10

Chapter 10: Markov Chains

In this chapter, you will learn to:

1. *Write transition matrices for Markov Chain problems.*
2. *Explore some ways in which Markov Chains models are used in business, finance, public health and other fields of application*
3. *Find the long term trend for a Regular Markov Chain.*
4. *Solve and interpret Absorbing Markov Chains.*

10.1 Introduction to Markov Chains

In this chapter, you will learn to:

1. *Write transition matrices for Markov Chain problems.*
2. *Use the transition matrix and the initial state vector to find the state vector that gives the distribution after a specified number of transitions.*

We will now study stochastic processes, experiments in which the outcomes of events depend on the previous outcomes; stochastic processes involve random outcomes that can be described by probabilities. Such a process or experiment is called a **Markov Chain** or **Markov process**. The process was first studied by a Russian mathematician named Andrei A. Markov in the early 1900s.

About 600 cities worldwide have bike share programs. Typically a person pays a fee to join a the program and can borrow a bicycle from any bike share station and then can return it to the same or another system. Each day, the distribution of bikes at the stations changes, as the bikes get returned to different stations from where they are borrowed.

For simplicity, let's consider a very simple bike share program with only 3 stations: A, B, C. Suppose that all bicycles must be returned to the station at the end of the day, so that each day there is a time, let's say midnight, that all bikes are at some station, and we can examine all the stations at this time of day, every day. We want to model the movement of bikes from midnight of a given day to midnight of the next day. We find that over a 1 day period,

- of the bikes borrowed from station A, 30% are returned to station A, 50% end up at station B, and 20% end up at station C.
- of the bikes borrowed from station B, 10% end up at station A, 60% have been returned to station B, and 30% end up at station C
- of the bikes borrowed from station C, 10% end up at station A, 10% end up at station B, and 80% are returned to station C.

We can draw an arrow diagram to show this. The arrows indicate the station where the bicycle was started, called its initial state, and the stations at which it might be located one day later, called the terminal states. The numbers on the arrows show the probability for being in each of the indicated terminal states.