

Error Propagation

The analysis of uncertainties (errors) in measurements and calculations is essential in the physics laboratory. We will start with a review of differentials from calculus, introduce partial derivatives, and then derive the general equation for error propagation.

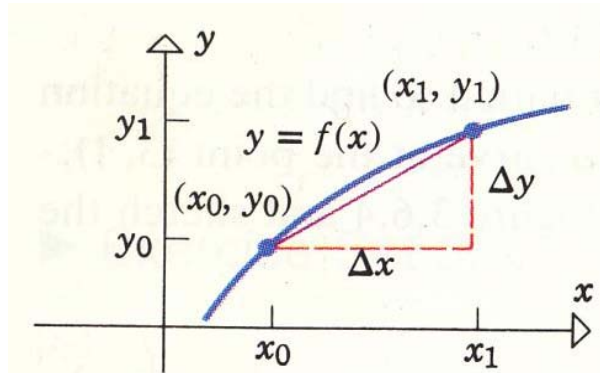
Review From Calculus

If $y = f(x)$, and x changes from some initial value x_0 to a final value x_1 , then there is corresponding change in y from $y_0 = f(x_0)$ to $y_1 = f(x_1)$. Thus, the increment

$$\Delta x = x_1 - x_0 \quad (1)$$

produces a corresponding increment

$$\Delta y = y_1 - y_0 = f(x_1) - f(x_0) \quad (2)$$

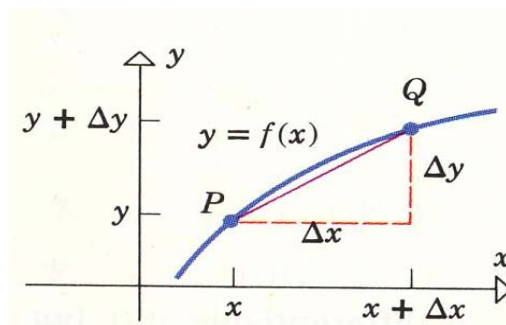


From equation (1) we have $x_1 = x_0 + \Delta x$ and thus (2) can be written as

$$\Delta y = f(x_0 + \Delta x) - f(x_0) \quad (3)$$

We will drop the zero subscript on x_0 and use x to denote some initial position as well as the variable. Equation (3) now becomes

$$\Delta y = f(x + \Delta x) - f(x) \quad (4)$$



The derivative can now be defined in the following equivalent forms:

$$\frac{dy}{dx} = \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x}$$

$$\frac{dy}{dx} = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

$$f'(x) = \lim_{\Delta x \rightarrow 0} \frac{\Delta f}{\Delta x}$$

Differentials

The symbol $\frac{dy}{dx}$ is the symbol for the derivative. Up to now the symbols dy and dx , called differentials, have no meaning by themselves. However, we shall now define them so that they do have a meaning.

Let's suppose that x is fixed and define dx to be an independent variable than can be assigned any value. If $f(x)$ is differentiable at x , we define by the formula

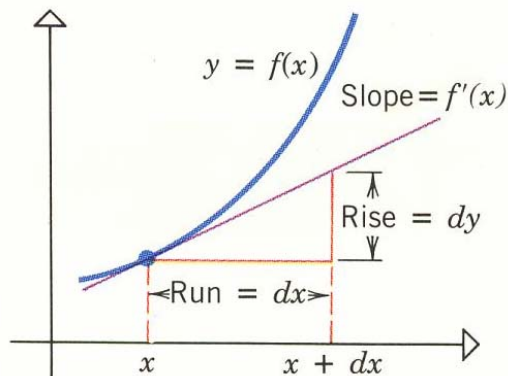
$$dy = f'(x)dx \quad (5)$$

If $dx \neq 0$, then we can divide both sides to obtain

$$\frac{dy}{dx} = f'(x) \quad (6)$$

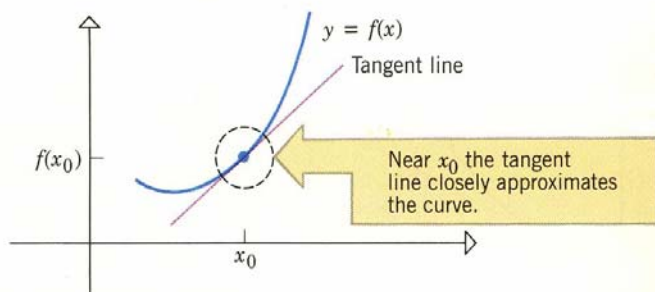
But $f'(x) = \frac{dy}{dx}$ is equal to the slope of the tangent line to $f(x)$ at x . Therefore, dy and dx correspond to the rise and run of this tangent line.

In general, the increment Δy and the differential dy are different. To see the difference let's consider the case when $dx = \Delta x$ and look at the tangent line to $y = f(x)$ at x .



- a) Δy is the change in y that occurs if we start at x and move **along the curve** until we have moved a distance Δx .
- b) dy is the change in y that occurs if we start at x and move **along the tangent line** until we have moved a distance Δx .

Consider the function $f(x)$ evaluated at the point x_0 . The figure below shows that for x close to x_0 , the tangent line to the curve at x_0 is a good approximation to the curve $f(x)$.



The tangent line goes through the point $(x_0, f(x_0))$ and has slope $f'(x_0)$. Using the point-slope form of a line for the tangent line,

$$y - y_0 = m(x - x_0) \quad \text{or} \quad y = y_0 + m(x - x_0)$$

Since we are considering x close to x_0 , then the height y of this tangent line will closely approximate the height $f(x)$ of the curve and then $y \approx f(x)$. Thus,

$$f(x) \approx f(x_0) + f'(x_0) \Delta x \quad (7)$$

But $\Delta x = x - x_0$ implies that $x = \Delta x + x_0$, thus

$$\boxed{f(x_0 + \Delta x) \approx f(x_0) + f'(x_0) \Delta x} \quad (8)$$

This is called the linear approximation of $f(x)$ near x when Δx is small.

Example - Use equation (8) to approximate $\sqrt{9.1}$

$$f(x) = \sqrt{x}$$

$$x = 9.1$$

$$x_o = 9.0$$

$$\Delta x = x - x_o = 0.1$$

$$f(9.1) \approx f(9.0) + f'(9.0)(0.1)$$

$$f(9.0) = \sqrt{9.0} = 3$$

$$f'(x) = \frac{1}{2\sqrt{x}}$$

$$f'(9) = \frac{1}{2\sqrt{9}} = 0.167$$

$$f(9.1) \approx 3.0167$$

This gives a 0.0026% error when comparing to the true value of $\sqrt{9.1} = 3.0166$.

Going back to Eq. (8) and rearranging gives the following,

$$\begin{aligned} f(x_o + \Delta x) &\approx f(x_o) + f'(x_o) \Delta x \\ f(x_o + \Delta x) - f(x_o) &\approx f'(x_o) \Delta x \\ \Delta y &\approx f'(x_o) \Delta x \end{aligned}$$

If we drop the subscript on x_o

$$\Delta y \approx f'(x) \Delta x \quad (9)$$

Now recall that,

$$dy = f'(x) dx \quad (10)$$

Now comparing Eq. (9) and Eq. (10) we see that if $\Delta x = dx$, then

$$\Delta y = dy \quad (11)$$

Thus, if Δx is relative “small” and we ensure that $\Delta x = dx$, then to a very good approximation

$$dy = f'(x) dx$$

can be written in the following form $\Delta y = \left(\frac{df}{dx}\right)\Delta x$. Using f instead of y :

$$\boxed{\Delta f = \left(\frac{df}{dx}\right)\Delta x} \quad (12)$$

This equation allows us to determine the error (uncertainty) in the function f associated with the error (uncertainty) Δx in the physical measurement x . In this way we say that the error Δx propagates to produce an error Δf in the calculated value of f . We refer to this as propagation of error. Equation (12) is valid only for functions of one variable. Later we will generalize this equation to include functions of more than one variable.

Example – A sphere has radius $r = 50.00 \text{ cm} \pm 0.02 \text{ cm}$. Calculate the uncertainty in the volume.

$$\begin{aligned}\Delta V &= \left(\frac{dV}{dr}\right)\Delta r \\ V &= \frac{4}{3}\pi r^3 \\ \frac{dV}{dr} &= 4\pi r^2 \\ \Delta V &= (4\pi r^2)\Delta r \\ \Delta V &= 4\pi(50.00\text{cm})^2(0.02\text{cm}) \\ \Delta V &= 600\text{cm}^3 \\ V &= 5.236 \times 10^5 \pm 600 \text{ cm}^3\end{aligned}$$

Partial Derivatives

There are many formulas in which a function depends on two or more variables. For example, the volume of a box of lengths b , w , and h

$$V = bwh$$

Likewise, the function $z = f(x,y)$ is a function of two independent variable x and y . As an example

$$z = f(x,y) = x^2 + 3xy$$

is a function of the independent variable x and y .

Let $f = f(x,y)$. If we hold y constant, then $f(x,y)$ is a function of x alone. The derivative of $f(x,y)$ with respect to x is denoted by $\frac{\partial f}{\partial x}$ and is called the partial derivative of f with respect to x . Likewise, we could have fixed x and taken the partial derivative of f with respect to y . This would be denoted by $\frac{\partial f}{\partial y}$. Using the above function

$$z = x^2 + 3xy$$

$$\frac{\partial z}{\partial x} = 2x + 3y$$

$$\frac{\partial z}{\partial y} = 3x$$

Example Let $f = 4xyz + x^2 + 3yz^3$

$$\frac{\partial f}{\partial x} = 4yz + 2x$$

$$\frac{\partial f}{\partial y} = 4xz + 3z^3$$

$$\frac{\partial f}{\partial z} = 4xy + 9yz^2$$

Propagation of Errors

For a function of one variable we've shown that

$$\Delta f = \left(\frac{df}{dx} \right) \Delta x .$$

Extending this equation to a function of 3 variables $f = f(x,y,z)$ the result is

$$\Delta f = \left(\frac{\partial f}{\partial x} \right) \Delta x + \left(\frac{\partial f}{\partial y} \right) \Delta y + \left(\frac{\partial f}{\partial z} \right) \Delta z$$

where the derivatives are partial derivatives. I have stated this result without proof but the proof is similar to the derivation of $\Delta f = \left(\frac{df}{dx} \right) \Delta x$.

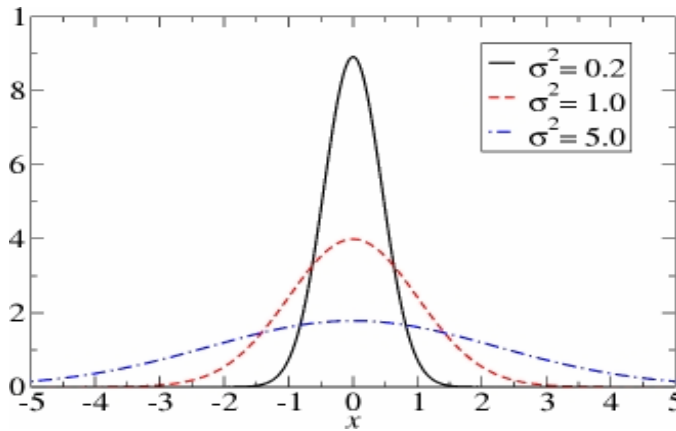
Suppose you make a series of measurements determining sets of values for the variables x , y , and z with uncertainties σ_x , σ_y , and σ_z . We will now show how to calculate the uncertainty in $f = f(x,y,z)$ produced by such uncertainties.

If the uncertainties of the measurements are random and independent (uncertainties must also be “small”), then after repeated measurements the measurements will be distributed (centered) about the mean. The distribution that results is called a Gaussian or Normal Distribution (bell-shaped curve) which is given by the following formula:

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\bar{x})^2}{2\sigma^2}}$$

σ = standard deviation
 \bar{x} = mean

The following are graphs of the Normal Distribution equation with different values of the standard deviation σ .



This function has very important properties in probability and statistics. For our purpose we will focus on the representation of the quantity σ which is called the standard deviation. The standard deviation is defined by the following statistical expression

$$\sigma = \sqrt{\frac{\sum_{i=1}^n (X_i - \bar{X})^2}{n}}$$

Some of the properties of the standard deviation are the following:

1. Measure of the average deviation from the mean.
2. Measure of the dispersion of a set of data from the mean.
3. 68% of data falls within one standard deviation.
4. 68% probability that a measurement falls within one standard deviation.

5. Small σ corresponds to a sharp Gaussian Distribution and a large σ corresponds to a broad distribution.

The standard deviation σ can also be written as

$$\sigma = \sqrt{\frac{\sum (\Delta x)^2}{n}}$$

where $\Delta x = x_i - \bar{x}$ is the deviation of a single measurement from the mean. We shall now define the standard deviation σ to be equal to the uncertainty of a single measurement.

The uncertainty in the function $f = f(x,y,z)$ is given by

$$\sigma_f = \sqrt{\frac{\sum (\Delta f)^2}{n}} \quad (13)$$

where Δf is given by

$$\Delta f = \left(\frac{\partial f}{\partial x}\right) \Delta x + \left(\frac{\partial f}{\partial y}\right) \Delta y + \left(\frac{\partial f}{\partial z}\right) \Delta z \quad (14)$$

Substituting Eq. (14) into Eq. (13) gives

$$\begin{aligned} \sigma_f^2 &= \frac{\sum(\Delta f)}{n} \\ \sigma_f^2 &= \frac{1}{n} \sum \left[\left(\frac{\partial f}{\partial x} \right) \Delta x + \left(\frac{\partial f}{\partial y} \right) \Delta y + \left(\frac{\partial f}{\partial z} \right) \Delta z \right]^2 \\ \sigma_f^2 &= \frac{1}{n} \sum \left\{ \left[\left(\frac{\partial f}{\partial x} \right) \Delta x + \left(\frac{\partial f}{\partial y} \right) \Delta y + \left(\frac{\partial f}{\partial z} \right) \Delta z \right] \left[\left(\frac{\partial f}{\partial x} \right) \Delta x + \left(\frac{\partial f}{\partial y} \right) \Delta y + \left(\frac{\partial f}{\partial z} \right) \Delta z \right] \right\} \\ \sigma_f^2 &= \frac{1}{n} \sum \left[\begin{aligned} &\left(\frac{\partial f}{\partial x} \right)^2 (\Delta x)^2 + \left(\frac{\partial f}{\partial y} \right)^2 (\Delta y)^2 + \left(\frac{\partial f}{\partial z} \right)^2 (\Delta z)^2 + 2 \left(\frac{\partial f}{\partial x} \right) \left(\frac{\partial f}{\partial y} \right) \Delta x \Delta y + 2 \left(\frac{\partial f}{\partial x} \right) \left(\frac{\partial f}{\partial z} \right) \Delta x \Delta z + \\ &2 \left(\frac{\partial f}{\partial y} \right) \left(\frac{\partial f}{\partial z} \right) \Delta y \Delta z \end{aligned} \right] \\ \sigma_f^2 &= \left(\frac{\partial f}{\partial x} \right)^2 \sum \frac{(\Delta x)^2}{n} + \left(\frac{\partial f}{\partial y} \right)^2 \sum \frac{(\Delta y)^2}{n} + \left(\frac{\partial f}{\partial z} \right)^2 \sum \frac{(\Delta z)^2}{n} + 2 \left(\frac{\partial f}{\partial x} \right) \left(\frac{\partial f}{\partial y} \right) \frac{1}{n} \sum \Delta x \Delta y \\ &+ 2 \left(\frac{\partial f}{\partial x} \right) \left(\frac{\partial f}{\partial z} \right) \frac{1}{n} \sum \Delta x \Delta z + 2 \left(\frac{\partial f}{\partial y} \right) \left(\frac{\partial f}{\partial z} \right) \frac{1}{n} \sum \Delta y \Delta z \end{aligned}$$

Since the uncertainties in the measurement x, y, and z are random and independent, then

$$\begin{aligned} \sum \Delta x \Delta y &= 0 \\ \sum \Delta x \Delta z &= 0 \\ \sum \Delta y \Delta z &= 0 \end{aligned}$$

Therefore,

$$\sigma_f^2 = \left(\frac{\partial f}{\partial x} \right)^2 \sigma_x^2 + \left(\frac{\partial f}{\partial y} \right)^2 \sigma_y^2 + \left(\frac{\partial f}{\partial z} \right)^2 \sigma_z^2$$

$$\sigma_f = \sqrt{\left(\frac{\partial f}{\partial x} \right)^2 \sigma_x^2 + \left(\frac{\partial f}{\partial y} \right)^2 \sigma_y^2 + \left(\frac{\partial f}{\partial z} \right)^2 \sigma_z^2}$$

General Error Propagation Equation

Example $f = x + y$

$$\frac{\partial f}{\partial x} = 1$$

$$\frac{\partial f}{\partial y} = 1$$

$$\sigma_f = \sqrt{\sigma_x^2 + \sigma_y^2}$$

Example

$$f = xy$$

$$\frac{\partial f}{\partial x} = y$$

$$\frac{\partial f}{\partial y} = x$$

$$\sigma_f = \sqrt{y^2 \sigma_x^2 + x^2 \sigma_y^2}$$

$$\frac{\sigma_f}{xy} = \sqrt{\frac{y^2 \sigma_x^2}{x^2 y^2} + \frac{x^2 \sigma_y^2}{x^2 y^2}}$$

$$\sigma_f = f \sqrt{\left(\frac{\sigma_x}{x}\right)^2 + \left(\frac{\sigma_y}{y}\right)^2}$$

Example

Calculate the value of z and the uncertainty if $z = wx + y^2$

$$w = 4.52 \pm 0.02 \text{ cm}$$

$$x = 2.0 \pm 0.2 \text{ cm}$$

$$y = 3.0 \pm 0.6 \text{ cm}$$

$$z = z_{best} \pm \sigma_z$$

$$\sigma_z = \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 \sigma_x^2 + \left(\frac{\partial z}{\partial y}\right)^2 \sigma_y^2 + \left(\frac{\partial z}{\partial w}\right)^2 \sigma_w^2}$$

$$\frac{\partial z}{\partial x} = w = 4.52 \text{ cm}$$

$$\frac{\partial z}{\partial y} = 2y = 6.0 \text{ cm}$$

$$\frac{\partial z}{\partial w} = x = 2.0 \text{ cm}$$

$$\sigma_z = \sqrt{(4.52 \text{ cm})^2 (0.2 \text{ cm})^2 + (6.0 \text{ cm})^2 (0.6 \text{ cm})^2 + (2.0 \text{ cm})^2 (0.02 \text{ cm})^2}$$

$$\sigma_z = 3.7 \text{ cm}^2 \approx 4 \text{ cm}^2$$

$$z_{best} = w_{best} x_{best} + y_{best}^2$$

$$z_{best} = (4.52 \text{ cm})(2.0 \text{ cm}) + (3.0 \text{ cm})^2 = 18.0400 \text{ cm}^2 \approx 18 \text{ cm}^2$$

$$z_{best} = 18 \pm 4 \text{ cm}^2$$